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Logarithms

Warning: Logarithms rarely show up on the exam. This chapter is only for those who want to cover all their bases (haha!).

At this point, we know how to solve an equation like

$$x^3 = 8$$

We take the cube root of both sides to get $x = 2$. But what about

$$3^x = 8$$

Do we take the xth root of both sides? That doesn't really work. Instead, we have to use logs (short for logarithms).

Here's how they work. Given an equation like $3^x = 8$, we can isolate x like so:

$$x = \log_3 8$$

The little "3" is called the base. The "8" is called the argument or the power. Notice where those two numbers came from and where they were placed. It's important that you know how to go from one form to the other:

$$3^x = 8 \qquad\qquad x = \log_3 8$$

Now to evaluate $\log_3 8$, we need a calculator, which would give you

$$x = \log_3 8 \approx 1.89$$

As you can see, $\log_3 8$ is just a number. That's all it is. Don't be scared by it. Now we can state that

$$3^{1.89} = 8$$

We used a calculator for this one, but don't worry about any calculator steps. The ACT will never ask you to evaluate a logarithm that you can't do by hand.

Lastly, a log with a base of 10 is typically written without the base. For example, $\log 7 = \log_{10} 7$.

EXAMPLE 1: What is the value of $\log_2\left(\frac{1}{8}\right)$?

A. -3 **B.** $-\frac{1}{3}$ **C.** $\frac{1}{4}$ **D.** $\frac{1}{3}$ **E.** 3

Let $\log_2\left(\frac{1}{8}\right) = x$. Then by definition,

$$2^x = \frac{1}{8}$$

Since $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, $x = -3$. Answer $\boxed{(A)}$.

EXAMPLE 2: Given that $5^{2x} = 9$, which of the following gives the value of x ?

A. $\frac{\log_2 9}{5}$ **B.** $\frac{\log_5 9}{2}$ **C.** $2\log_5 9$ **D.** $5\log_2 9$ **E.** $9\log_2 5$

The equation $5^{2x} = 9$ is equivalent to

$$2x = \log_5 9$$

$$x = \frac{\log_5 9}{2}$$

Answer $\boxed{(B)}$.

Laws of Logarithms

There are several laws of logarithms you should know:

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(x^y) = y\log_a x$
- $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a b = \frac{\log_c b}{\log_c a}$

The first two don't really need to be memorized because they stem from the basic definition of a logarithm. You should memorize the rest.

The last one allows you to change the base of a logarithm. For example, if you wanted to convert $\log_2 5$ to a logarithm expression in base 10, which might be easier to input on your calculator, then $\log_2 5 = \frac{\log 5}{\log 2}$.

EXAMPLE 3: Which of the following is equivalent to $\log_3 5 + \log_3 7$?

A. $\log_3 12$ **B.** $\log_3 35$ **C.** $2\log_3 6$ **D.** 12^3 **E.** 3^{12}

$$\log_3 5 + \log_3 7 = \log_3(5 \cdot 7) = \log_3 35$$

Answer $\boxed{(B)}$.

EXAMPLE 4: If $p = \log_2 x$, what is the value of $\log_2(2x^3)$ in terms of p ?

A. $6p$ **B.** $2p^3$ **C.** $1+3p$ **D.** $3+3p$ **E.** $1+p^3$

$$\log_2(2x^3) = \log_2 2 + \log_2 x^3 = 1 + 3\log_2 x = 1 + 3p$$

Answer $\boxed{(C)}$. Note that we cannot move the exponent "3" to the front first because it applies only to x, not the entire argument.

EXAMPLE 5: If $\log_3 b - \log_3 4 = 2$, then $b = ?$

A. 6 **B.** 8 **C.** 13 **D.** 24 **E.** 36

$$\log_3 b - \log_3 4 = 2$$
$$\log_3\left(\frac{b}{4}\right) = 2$$

By definition,

$$3^2 = \frac{b}{4}$$
$$b = 3^2 \cdot 4 = 36$$

Answer $\boxed{(E)}$.

CHAPTER EXERCISE: Answers for this chapter start on page 262.

1. What is the value of x if $\log_4 x = 3$?

   A. $\sqrt[4]{3}$
   B. $\sqrt[3]{4}$
   C. 12
   D. 64
   E. 81

2. If $a = \log_5 3$ and $b = \log_5 4$, which of the following expressions is equal to 12?

   A. ab
   B. $a + b$
   C. $5^a + 5^b$
   D. 5^{a+b}
   E. 25^{ab}

3. If c is a positive number such that $\log_c 9 = \frac{1}{2}$, then $c = ?$

   A. $\frac{1}{3}$
   B. $\sqrt{3}$
   C. 3
   D. 18
   E. 81

4. If $\log_5 4 = x$, what is the value of 5^{2-x} ?

   A. $\frac{1}{25}$
   B. $\frac{4}{25}$
   C. $\frac{25}{4}$
   D. 21
   E. 25

5. Given that $\log_3 x^2 = c$, what is the value of $\log_3 x$ in terms of c ?

   A. $\frac{c}{2}$
   B. $c - 2$
   C. $2c$
   D. c^2
   E. $\sqrt{c}$

6. If $\log_2 7 = p$ and $\log_2 3 = q$, what is the value of $\log_2 63$ in terms of p and q ?

   A. $3pq$
   B. $p + 2q$
   C. $p + 3q$
   D. $3p + 3q$
   E. $p + q^2$

7. For all $x > 0$, which of the following expressions is equivalent to $\log\left((3x)^2\right)$?

   A. $\log 3 + 2\log x$
   B. $2\log 3 + \log x$
   C. $2\log 3 + 2\log x$
   D. $2(\log 3)(\log x)$
   E. $\log 6 + \log 2x$

8. If $a, b,$ and c are positive numbers such that $a^x = b$ and $a^y = c$, then $xy = ?$

   A. $\sqrt[a]{bc}$
   B. $\log_a(bc)$
   C. $\log_b a + \log_c a$
   D. $(\log_a b)(\log_a c)$
   E. $(\log_b a)(\log_c a)$

9. If $x > 0$ and $\log_4(x+3) + \log_4(x-3) = 2$, then $x = ?$

   A. 5
   B. $\sqrt{7}$
   C. 8
   D. $\sqrt{17}$
   E. 25

10. If $\log_b 40 - \log_b 5 = 3$, then $b = ?$

    A. 2
    B. 5
    C. 8
    D. 15
    E. 512